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Localization of gravity in brane world cosmologies

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The most remarkable and interesting feature of brane world scenario is the use of bulk's curvature to localize gravity on the brane, albeit with fine tuning of the brane and bulk parameters. For FRW expanding universe on the brane, it is a moving hypersurface in Schwarzschild anti de Sitter bulk spacetime. We show that zero mass gravitons have bound state on the brane for suitable values of brane and bulk parameters. There occur various cases giving rise to different cosmological models, in particular we discuss the case of inflationary model with positive cosmological constant.

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The view that our Universe might actually have dimensions more than four is anchored on the recent developments in string and M-theories in which gravity arises as a truly higher dimensional interaction. Only in the low energy limit it manifests in the familiar 4-dimensional general relativity (GR). This has initiated a lot of activity in recent times. Though the basic idea was already there in the form of Kaluza-Klein (KK) theories, the recent spurt is primarily due to the possibility which helps solve the mass hierarchy problem in the standard model of particle physics.

In some of these models [1] the extra dimensions can be as large as millimeter which is however less than the current observational limits on low scale gravity. Notable are the models in which the extra dimensions are allowed to be of infinite extent [2, 3]. These models have Z_2-symmetry, which is motivated by the reduction of M theory to $E_8 \times E_8$ heterotic string theory [4]. The single brane Randall-Sundrum (RS) model [3] has attracted a lot of interest and activity. In this model the Minkowski flat brane in 5-D anti de Sitter (AdS) bulk has a positive tension. It is then possible to recover Newton's inverse square law with r^{-4} correction term which arises from massive KK modes contribution. There have been various generalizations of this [5] in the form of thick branes [6], AdS branes [7] and brane models without Z_2 symmetry [8].

In the overall view of the brane world scenario, all matter fields live on the 3-brane which is the 4-D physical Universe while gravity can propagate in the extra dimensions, say 5-D bulk. The bulk and brane spacetimes are joined together through the Israel junction conditions [9]. Since, the standard Einstein equations on the brane are modified by the bulk effects it opens a whole new vista for investigation of astrophysical and cosmological consequences of these models (see for eg. [10]). The connection with CFT correspondence has also been studied (see for eg. [11]). For complete solution of the problem, one has to solve the Λ-vacuum equation in 5-D for the bulk spacetime and simultaneously the modified Einstein equation which in addition to the square of stress tensor also contains the projection of the bulk Weyl curvature on the brane (dark radiation) and then the two solutions should be joined together with the Israel junction conditions [9]. It is by all means a very formidable task and it is therefore no surprise that there exist only few complete solutions to date. The two important cases are flat/vacuum brane with AdS bulk and FRW brane with Schwarzschild-AdS (S-AdS) bulk [12, 13, 14].

In S-AdS bulk, the brane will in general be moving unless the parameters are properly fine tuned (eg. for the RS case $\sigma = (3/4\pi G_5 l), \Lambda_4 = 0$, where σ is the brane tension, G_5 is the 5-D gravitational constant and l is the radius of curvature of the bulk spacetime). Further static brane cannot harbour expansion which is essential for realistic cosmological models. A slight off tuned value of σ or non zero value of black hole mass would set it moving [13].

Though localization of gravity for the AdS bulk with flat brane and for some generalizations of it has been established [7, 15, 16], it has not yet been investigated for the S-AdS bulk with FRW brane. The problem becomes difficult if one notes that unlike the RS case, the brane is dynamic in the static bulk and it is non trivial to fix the boundary conditions on the modes. For localizability of gravity on the brane, first there should exist bound normalizable mode for zero mass graviton and plus there should also exist KK modes to give the correction to the Newtonian gravity. All this could be studied by considering perturbation of the bulk metric and the brane motion determined by the Israel junction conditions. This is what we shall concern ourselves in this investigation which should be quite indicative of localization of gravity on the brane. For actual demonstration of recovering Newtonian gravity with the correction one has to do propagator analysis which is hard to carry out in a non-static spacetime and we shall not attempt that here.

It is well known that localizability is very sensitive to fine tuning of parameters. A priori, there is no well established criterion to check this. For instance there does exist a bulk spacetime, which is an exact solution of the

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Λ-vacuum equation, for which gravity is non localizable on the brane [17]. This is the case of Nariai metric which has non zero Weyl curvature. Note that Weyl is non zero for S-AdS as well. It is therefore pertinent to find under what conditions do zero mass gravitons have bound state on FRW brane? This is the most critical question for brane world cosmologies and that is what we wish to address in this letter.

Since our brane is a hypersurface around the black hole, any movement of the brane towards or away from the black hole would be interpreted as contraction or expansion by the observers on the brane. That is how expansion is directly related to motion of the brane. To see whether zero mass graviton has a bound normalizable state in these cosmological models not only one has to solve for the graviton perturbation equation but also must take into account the brane trajectory in the bulk, governed by Israel matching conditions. We shall now study metric perturbations of S-AdS bulk and shall first recover RS flat brane results. Then we shall in particular consider the case of $k=1, \Lambda_4>0$, while a comprehensive discussion of all possible cases would be done elsewhere [18].

In the five dimensional bulk we have the S-AdS metric,

$$ds^2 = -e^{2\beta}\,dt^2 + e^{-2\beta}dy^2 + y^2\left[\frac{dr^2}{1-kr^2} + r^2 d\Omega_2^2\right] \quad (1)$$

where

$$e^{2\beta} = \left(k + \frac{y^2}{l^2} - \frac{M}{y^2}\right). \quad (2)$$

Here $k=0,\pm1$ is the curvature index and M is the mass parameter of the bulk black hole.

The static limit would be given by $g_{00}=0$ which leads to,

$$y_h^2 = \frac{l^2}{2}\left(-k+\sqrt{k^2+\frac{4M}{l^2}}\right) \quad (3)$$

and since the metric is spherically symmetric this also defines the location of event horizon of the spacetime. The above metric is the solution of the Λ-vacuum equation,

$$G_{ab} = -\Lambda_5\, g_{ab}, \qquad \Lambda_5 = -6/l^2. \quad (4)$$

The Latin indices which label the bulk spacetime (x^μ, y) run from 0...4 and the Greek indices labelling brane spacetime (x^μ) would run from 0...3. We consider the metric perturbations of the above metric $g_{ab}^{(B)}$, i.e. $h_{ab} = g_{ab} - g_{ab}^{(B)}$, and obtain in the standard way the following wave equation

$$\Box_5 h_{ab} = \Lambda_5\, h_{ab}. \quad (5)$$

Our interest lies in finding out the behavior of gravitons on the brane due to the presence of extra dimension. For that we would take the metric perturbations in the extra dimension to vanish, $h_{t\,y} = h_{r\,y} = h_{\theta\,y} = h_{\phi\,y} = h_{y\,y} = 0$. We would further impose the transverse-traceless gauge conditions, $\nabla^\mu h_{\mu\nu} = 0$, $h^\mu{}_\mu = 0$ and work with the ansatz that $h_{ab}(x^\mu, y) = h_{ab}(x^\mu)\,\Psi(y)$. Substituting this in eq.(5) and using m^2 as the constant of separation of variables, the y dependence turns out to be

$$\left(\frac{y^2}{l^2} - \frac{M}{y^2} + k\right)\Psi'' + \left(\frac{3M}{y^3} - \frac{k}{y} + \frac{y}{l^2}\right)\Psi' - \frac{2M}{y^4}\Psi + m^2\Psi = 0 \quad (6)$$

where prime denotes a derivative with respect to y. This equation can be written down in the form

$$\Psi'' + a_1(y)\Psi' + a_2(y)\Psi = 0 \quad (7)$$

which can be transformed into a wave equation form with the transformation $\Psi(y) = \phi(y)\psi(y)$ where $\phi(y)$ is such that it eliminates the first order derivative term and hence we get the desired wave equation

$$-\frac{1}{2}\psi(y)'' + V\psi(y) = m^2\psi(y). \quad (8)$$

This equation holds necessary information about the effect of extra dimension on gravitons on the brane. The potential consists of two parts, $V = V_f + V_m$, which are given by

$$\begin{aligned} V_f = &- \frac{6l^2(5My^4 - ky^6)}{8(y^5 + l^2(-My + ky^3))^2} + \frac{Ml^2}{y^2(y^4 - Ml^2 + kl^2y^2)} \\ &- \frac{y^8 + l^4(-15M^2 + 22kMy^2 - 3k^2y^4)}{8(y^5 + l^2(-My + ky^3))^2} \end{aligned} \quad (9)$$

and the interaction part

$$V_m = -\frac{m^2l^2y^2 - 2m^2(y^4 - Ml^2 + kl^2y^2)}{2(y^4 - Ml^2 + kl^2y^2)}. \quad (10)$$

Obviously, the shape of the potential will play determining role in localization of gravitons. The potential V for all the cases provides a negative infinite well at the event horizon of the black hole of mass M, which holds irrespective of the matter distribution on the brane. The potential for the massless and massive modes for the case $k=1, M\neq0$ is shown in Fig. 1. The potential for the massless mode can be viewed as the $m\to0$ limit of the potential for massive modes for which $V>0$ for large y. The physically interesting branes would lie outside the horizon which would also serve as the high energy cutoff. The normalizibality of the massless mode depends on the aymptotic behavior of the potential and the form of the potential reflects that branes which are either static or ever expanding in the bulk would harbour localization.

The Israel junction conditions determine the relationship between the bulk and brane parameters,

$$\Lambda_4 = \frac{\Lambda_5}{2} + \frac{16\pi^2}{3}G_5^2\sigma^2,\; G_4 = \frac{4\pi}{3}G_5^2\,\sigma \quad (11)$$

and also the brane motion given by the equation (assuming Z_2 symmetry) [13],

$$\dot{y}^2 - \frac{8\pi G_4}{3}\rho\left(1+\frac{\rho}{2\sigma}\right)y^2 + (1-\sigma^2/\sigma_0^2)\frac{y^2}{l^2} - \frac{M}{y^2} + k = 0. \tag{12}$$

Here dot refers to derivative relative to proper time τ and ρ is the energy density on the brane. The y coordinate now plays a dual role. It not only tells us about the effect on graviton fluctuations due to extra dimension but also parameterizes the brane trajectory. As is clear from eq. (12) that motion of brane is determined by both black hole mass and energy distribution on the brane. The localization would therefore depend on energy distribution in both bulk and brane. The density term in the above equation would scale as $\rho \sim y^{-4}$ or $\rho \sim y^{-3}$ depending upon whether the universe is radiation or matter dominated. Thus at large y the behaviour of the brane trajectory is solely governed by the sign of the effective cosmological constant on the brane which then plays a determining role in the localization of the massless mode. Hence we shall set $\rho = 0$, for its inclusion will not disturb localization, and solve for the brane dynamics. Defining the critical tension to be $\sigma_0 = 3/4\pi G_5 l$ (which determines the sign of Λ_4) we get

$$\dot{y}^2 - \frac{\Lambda_4}{3}y^2 - \frac{M}{y^2} + k = 0. \tag{13}$$

Note that this equation is non linear in the highest order of derivative and hence it will not have unique solution.

Now we would see how the above metric perturbation equation and the brane dynamics equation yield RS model. For this $k = M = 0, \sigma = \sigma_0$, the metric eq.(1) takes the form

$$ds^2 = -\frac{y^2}{l^2}dt^2 + \frac{l^2}{y^2}dy^2 + y^2\left(dr^2 + r^2 dr^2 + r^2\sin^2\theta d\phi^2\right), \tag{14}$$

which by the transformation, $\eta = l\ln(l/y)$, reduces to the familiar RS form

$$ds^2 = d\eta^2 + e^{-2\eta/l}(-dt^2 + dx_1^2 + dx_2^2 + dx_3^2). \tag{15}$$

RS brane is static and located at $\eta = 0$, which means at $y = l$. It sits in the potential, $V = -1/8y^2 + m^2(1 - l^2/2y^2)$. The massless mode is bound and for the massive modes for which $m > 1/2l$, $V > 0$ and they are unbound. The form of the potential also suggests that there exist discrete modes for $m < 1/2l$. The massless mode is normalizable and since it is also bound, this leads to localization of gravity. The fact that V turns positive for massive modes gives the continuum spectrum and the RS correction to Newtonian potential.

For AdS branes the brane trajectory exhibits periodic motion, like for the case $k = -1, M = 0, \Lambda_4 < 0$ it is given by $y(\tau) = \sin(\sqrt{-\Lambda_4/3}\tau)/\sqrt{-\Lambda_4/3}$. The potential is similar as in RS case, apart from that it blows up at origin and there is an infinite negative well at $y = l$, but the brane trajectory is such that it does not yield the required behaviour of the potential for the ground state wavefunction to be normalizable. Hence the massless mode would not be localized for AdS brane (but that would be for dS brane because the solution of the brane motion equation does permit normalized wavefunction). We thus recover the well-known result for AdS bulk and AdS/dS brane [7]. The behavior of brane trajectory also plays critical role in localization of gravity on the brane.

Now we turn to FRW brane. Eq.(13) yields host of solutions for interesting cosmological scenarios including the inflationary solutions for all values of k with a positive Λ_4 on the brane, which is also favored by current observations of type Ia supernovae [19]. A detailed discussion of all these cases would be done elsewhere [18], here we would as a representative consider the case of $k = 1$, $M \neq 0$ and $\Lambda_4 > 0$.

The metric on the brane is FRW,

$$ds_4^2 = -d\tau^2 + y^2(\tau)\left(\frac{dr^2}{1-r^2} + d\Omega_2^2\right) \tag{16}$$

where $d\tau^2 = exp(2\beta(t))(1 - exp(-4\beta(t))(dy/dt)^2)dt^2$. Solving eq.(13) we get,

$$y(\tau) = \sqrt{\frac{3}{2\Lambda_4}}\left[1 + n\sinh(2x) - \cosh(2x)\right]^{1/2} \tag{17}$$

where $x = \sqrt{\Lambda_4/3}\,\tau$ and $n = 2\sqrt{M\Lambda_4/3}$. For $n > 1$ the brane emerges out of the event horizon, expanding from $y = 0$ at $\tau = 0$ like a white hole [20] and would expand for ever, exponentially for large x.

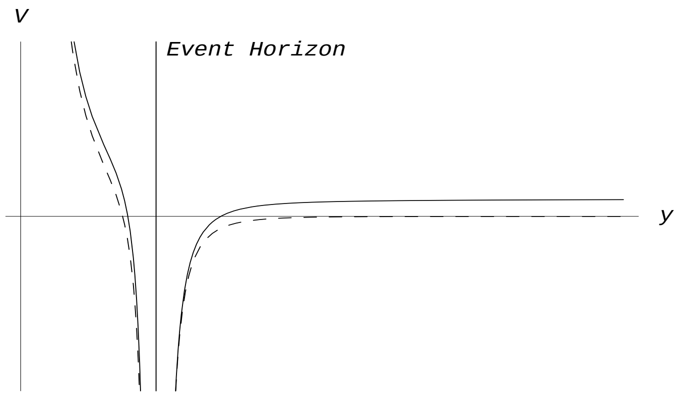

FIG. 1: Potential plot for $M \neq 0$ and $k = 1$. The dashed and dark lines indicate potential for massless and massive mode respectively.

The behavior of potential V for this case is shown in Fig.1. Note that V blows up at $y = 0$ and there is an infinite well at $y = y_h$, we shall therefore restrict to $y > y_h$. For the massless mode the potential profile favors boundedness and normalizability. The massive mode would be unbounded and would contribute to correction over Newtonian potential. To calculate the precise correction one should do a propagator analysis of graviton modes in this setting which we will not attempt here, however it can be shown that the massive mode as in RS case would be ml suppressed on the brane and hence an l^2/r^2 correction to the Newtonian potential is expected.

In order to show the correction suggested by massive modes we would first parameterize the location of the brane off the horizon by $\alpha = 1 - M/y^2$. Solving the Schroedinger equation, eq.(8), in the approximation $M \ll l^2$ and near the horizon ($y_h \sim \sqrt{M}$) we get,

$$\psi(y) = \sqrt{y}\,(C_1 I_{-\gamma/2}(\nu\, y) + C_2 I_{\gamma/2}(\nu\, y)) \tag{18}$$

where $\gamma = \sqrt{1 - 4l^4(1-2\alpha)(M-2\alpha l^2)/M^3}$ and $\nu = \sqrt{(1-\alpha)(M-2\alpha l^2)}ml/M$. Close to the horizon $\nu \sim ml/\sqrt{M}$ and since $y \sim \sqrt{M}$, the argument in the Bessel function as in the RS case is propotional to ml at the location of the brane. This suggests a l^2/r^2 correction to Newtonian potential due to massive modes. The constants C_1 and C_2 are to be determined by the boundary conditions which are to be fixed for a moving brane. However, the above equation holds only near the horizon. In the cosmological context we can view brane as a $t = constant$ hypersurface and fix the boundary conditions on $\psi(y)$ at the location of the brane and away from it, which determine $C_1 = (ml)^{(1+\gamma)/2} M^{\gamma/4}$ with $\gamma < 2$ and $C_2 = 0$.

The modifications to the standard GR would be most prevelant near the event horizon which is the high energy end and marks a cut off for the scale factor $y(\tau)$. As the brane moves out and expands, the potential as shown in Fig.1 becomes shallower and the high energy modifications die out with time. In particular, the RS correction to the Newtonian potential will die out as the universe expands. The discrete mode will turn into continuum with expansion which will lead to decrease in mass gap resulting into lowering of temperature. This augurs quite well with the common perception that the universe cools as it expands. This is a generic feature of brane world cosmological models.

The Hubble parameter for $k = 1$, $M \neq 0$ cosmological model is given by

$$H = \frac{\dot{y}}{y} = \sqrt{\frac{\Lambda_4}{3}}\,\frac{\Big[n\,\cosh(2x) - \sinh(2x)\Big]}{\Big[1 + n\,\sinh(2x) - \cosh(2x)\Big]}\,. \tag{19}$$

For large τ, it would imply exponential expansion leading to inflationary stage. From the curvature effects of the bulk S-AdS black hole, it could be envisioned in this case that a (dark) radiation universe is born on the brane.

The overall picture that emerges is that we could have different FRW cosmological models on the brane which are anchored onto S-AdS bulk. The curvature of the bulk spacetime and the brane trajectory are key players in the localization of gravity on the brane. The ultimate evolutionary fate of models is decided by the black hole mass and the cosmological constant on the brane. We have thus shown that S-AdS bulk does allow localization of gravity on FRW brane with $\Lambda_4 > 0$, which on taking into account cosmological observations can well serve as our physical universe.

In conclusion, we could say that brane cosmologies are firmly anchored in respect of localization of gravity. We have shown that the potential in the Schroedinger equation and the brane dynamics lead to bound state for zero mass graviton on the brane. Also there exist massive KK modes which are responsible for the high energy correction to the Newtonian potential. Even in the absence of actual propagator analysis, recovery of Newtonian gravity with correction term could be inferred so long as there exist bound normalizable zero mass graviton on the brane.

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